



Meshless Methods

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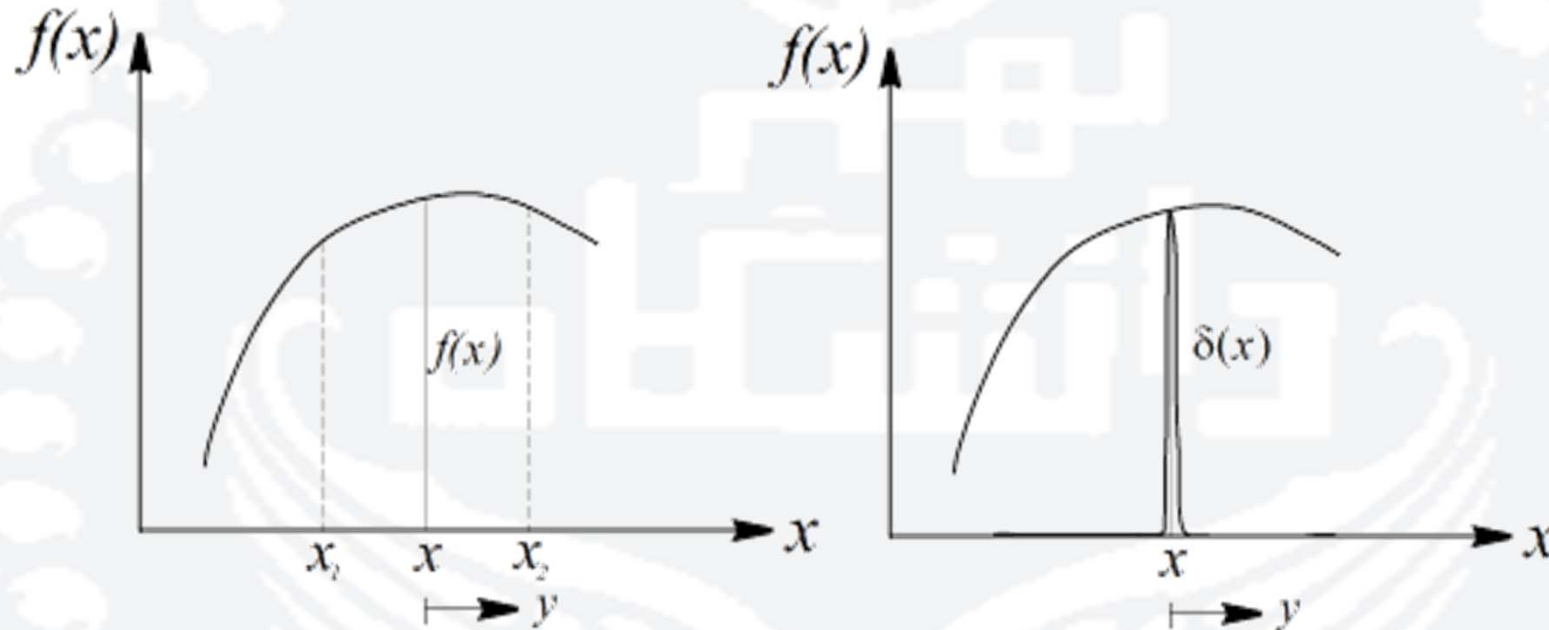
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Interpolation by Kernel Estimates:

1. SPH Method (1968)
2. Modified SPH Methods (1974-1976)
 1. Corrected SPH (1980)
 2. Reproducing Kernel Particle Method - RKPM
 3. SPH based on higher terms of Taylor Series

Interpolation by Kernel Estimates:

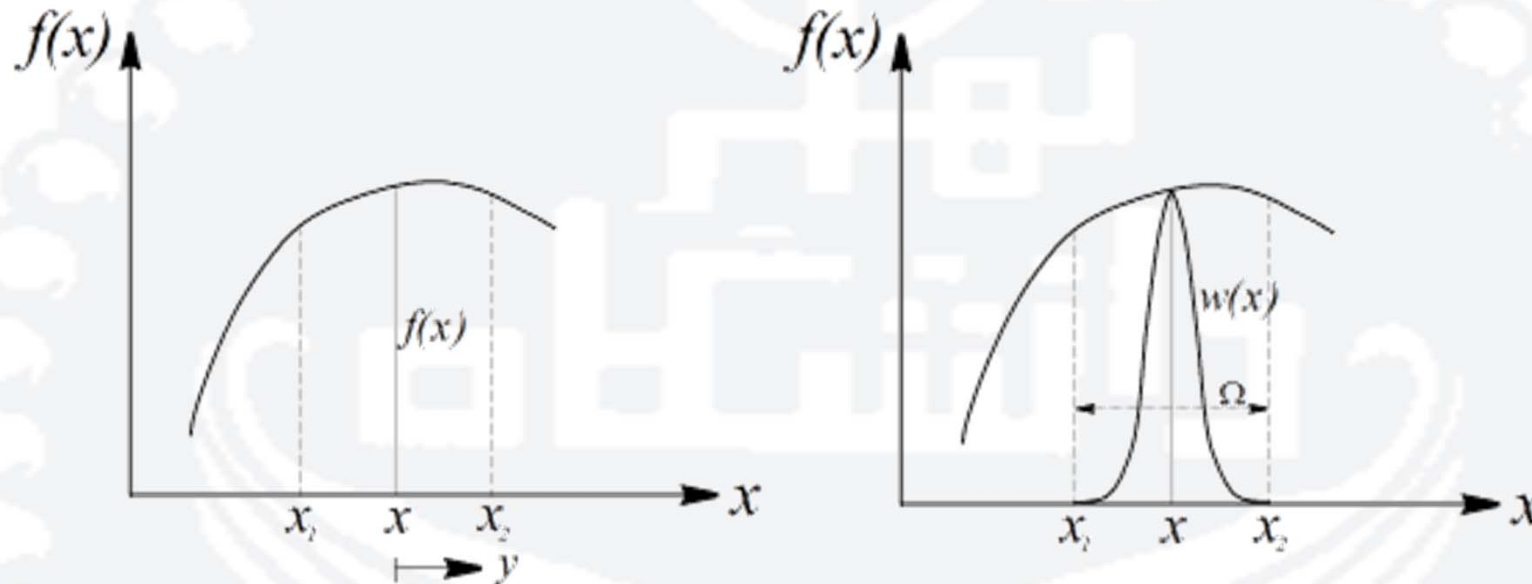


$$u(x) = \int_{-\infty}^{\infty} \delta(x - y) u(y) dy$$

$\delta(x)$ = dirac delta function

Exact but numerically difficult to use

Interpolation by Kernel Estimates:

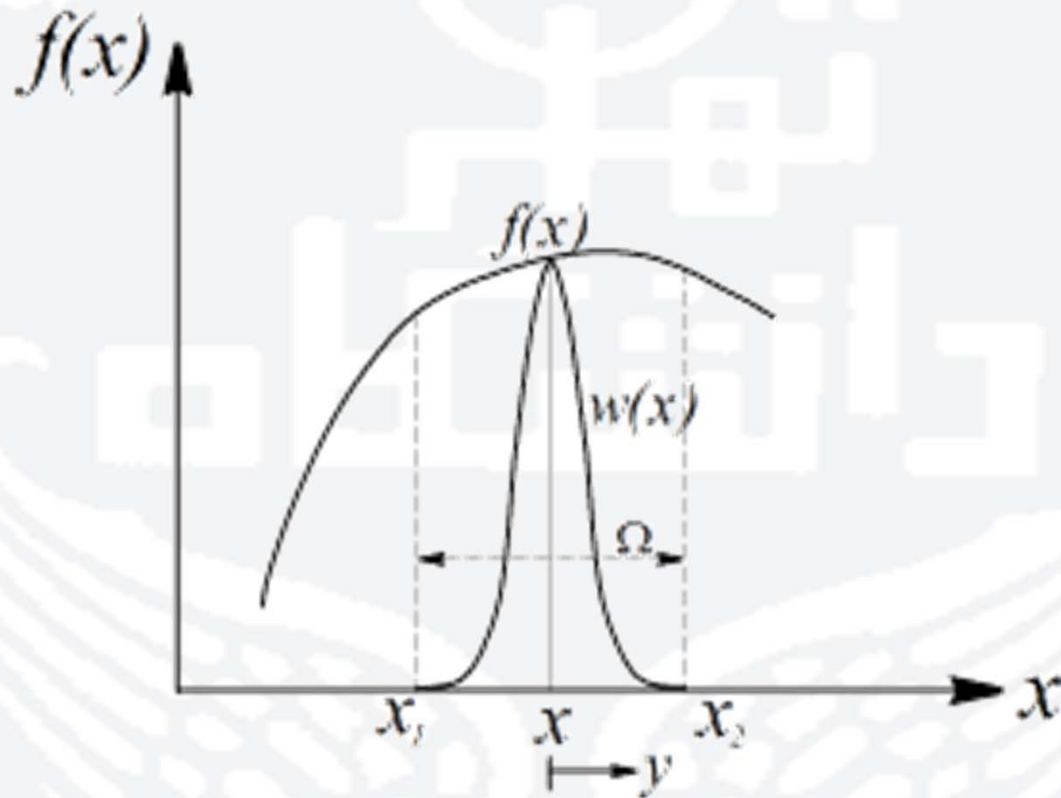


$$u_h(x) = \int_{\Omega} W(x-y, h) u(y) dy$$

$W(x-y, h) = W_h(x-y)$ = kernel/smoothing/weight function

h = the smoothing length, controlling the size of the compact support domain (smoothing or influence domain)

Interpolation by Kernel Estimates:



$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

Conditions of Weight Functions:

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

1. Positivity :

1. $W(x-y, h) > 0$ over Ω
2. Not necessary mathematically
3. To ensure meaningful or stable presentation of some physical phenomena
4. RKPM does not satisfy this condition

Conditions of Weight Functions:

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

2. Compact :

1. $W(x-y, h) = 0$ *outside* Ω
2. A requirement for the approximation to be generated from a local representation of nodes
3. $u_h(x)$ will depend only to the values of u at nodes (particles) in the smoothing domain
4. W and all its partial derivatives vanish outside a ball of radius h centred at x .
5. Some weight functions do not satisfy this condition.

Conditions of Weight Functions:

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

3. Unity :

1. $\int_{\Omega} W_h(x-y, h)dy = 1$
2. Assures the C^0 consistency of the integral form
3. Does not guarantee the C^0 consistency of the discrete form of the approximation
4. For nD : $\frac{1}{h^n} \int_{\Omega} W_h(x)dx = 1$ (at least in the limit when h goes to zero)

Conditions of Weight Functions:

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

4. Decay :

1. W is a monotonically decreasing function
2. A force exerted by a particle on another particle decreases with the increase of the distance between particles
3. Not a mathematical requirement
4. Not valid for RKPM

Conditions of Weight Functions:

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

5. Dirac delta function :

1. $W(s, h) = \delta(s)$ as $h \rightarrow 0$
2. If conditions (1-4) are satisfied , this condition is naturally satisfied.
3. h never goes to zero in practical computations.

Conditions of Weight Functions:

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

6. Differentiability :

1. $W_h(x) \in C^{\infty}(R^n)$
2. The smoothing kernel must be differentiable at least once. This is because the derivative of the kernel function should be continuous to prevent a large fluctuation in the force felt by the particle.

Kernel (Weight) Functions:

1- Cubic spline weight function (widely used)

$$W(x - x_I) = W(\bar{d}) = \begin{cases} \frac{2}{3} - 4\bar{d}^2 + 4\bar{d}^3 & \bar{d} < 0.5 \\ \frac{4}{3} - 4\bar{d} + 4\bar{d}^2 - \frac{4}{3}\bar{d}^3 & 0.5 \leq \bar{d} < 1 \\ 0 & \bar{d} > 1 \end{cases}$$

$$\bar{d} = \frac{\|x - x_I\|}{d_w} = \frac{d}{d_w} \quad \begin{array}{l} d_w \text{ is related to } h \text{ and can be different from point} \\ \text{to point, and defines the dimension of the domain} \\ \text{where } W \neq 0 \end{array}$$

- The weight function is not scaled and the unity condition is not satisfied.
- The first derivative of W is smooth.
- The second derivative of W is no longer smooth

Kernel (Weight) Functions:

2- Alternative cubic spline weight function

$$W_h(x) = \begin{cases} \frac{c}{h^d} \left(1 - \frac{3}{2}z^2 + \frac{3}{4}z^3\right) & z < 1 \\ \frac{c}{4h^d} (2-z)^3 & 1 \leq z < 2 \\ 0 & z > 2 \end{cases} \quad c = \begin{cases} \frac{2}{3} & : d = 1 \\ \frac{10}{7\pi} & : d = 2 \\ \frac{1}{\pi} & : d = 3 \end{cases}$$

$$z = \frac{\|x - y\|}{h}, \quad d = \text{dimension of the problem (1, 2 or 3)}$$

- The first derivative of W is smooth.
- The second derivative of W is no longer smooth

Kernel (Weight) Functions:

3- Quartic spline weight function

$$W(x - x_I) = W(\bar{d}) = \begin{cases} 1 - 6\bar{d}^2 + 8\bar{d}^3 - 3\bar{d}^4 & \bar{d} < 1 \\ 0 & \bar{d} \geq 1 \end{cases}$$
$$\bar{d} = \frac{\|x - x_I\|}{d_w} = \frac{d}{d_w}$$

d_w is related to h and can be different from point to point, and defines the dimension of the domain where

- The weight function is not scaled and the unity condition is not satisfied.
- The first derivative of W is smooth.
- Quartic spline weight function provides C^2 consistency, if Ω is fully located within the problem domain.

Kernel (Weight) Functions:

4- Alternative quartic spline weight function

$$W(x - x_I) = W(\bar{d}) = \begin{cases} \frac{2}{3} - \frac{9}{32} \bar{d}^2 + \frac{19}{192} \bar{d}^3 - \frac{5}{512} \bar{d}^4 & \bar{d} \leq 1 \\ 0 & \bar{d} \geq 1 \end{cases}$$

$\bar{d} = \frac{\|x - x_I\|}{d_w} = \frac{d}{d_w}$ d_w is related to h and can be different from point to point, and defines the dimension of the domain where

- The weight function is not scaled and the unity condition is not satisfied.
- The first and second derivatives of W are smooth.
- The second derivative of W does not equal to zero on the boundary.
- Quartic spline weight function provides C^2 consistency, if Ω is fully located within the problem domain.

Kernel (Weight) Functions:

5- Quintic spline weight function

$$W(r) = \frac{D(\lambda)}{h^\lambda} \begin{cases} (3-r)^5 - 6(2-r)^5 + 15(1-r)^5 & 0 \leq r < 1 \\ (3-r)^5 - 6(2-r)^5 & 1 \leq r < 2 \\ (3-r)^5 & 2 \leq r < 3 \\ 0 & 3 < r \end{cases}$$

$$r = \frac{\|\bar{x} - \bar{x}_I\|}{h} \quad \lambda = \text{dimension of the problem (1,2 or 3)}$$

$$D = \begin{cases} 120 & : \lambda = 1 \\ 7/478\pi & : \lambda = 2 \\ 3/359\pi & : \lambda = 3 \end{cases}$$

Kernel (Weight) Functions:

6- Exponential (Gaussian) weight function

$$W_h(x, y) = \frac{1}{\pi^{n/2} h^n} \exp\left(\frac{-\|x - y\|^2}{h^2}\right), 1 \leq n \leq 3$$

- h is the smoothing length

- Unity condition
- The first derivative of W .
- W does not vanish on the domain boundary.

$$\|x - y\| = h \rightarrow W_h(x, y) \neq 0$$

Kernel (Weight) Functions:

7- Alternative exponential weight function

$$W(x - x_I) = W(\bar{d}) = \begin{cases} \exp(-\bar{d}^2 / \alpha^2) & \bar{d} \leq 1 \\ 0 & \bar{d} > 1 \end{cases}$$

$$\alpha = \text{constant} \quad (\alpha = 0.3)$$

- The weight function is not scaled and the unity condition is not satisfied.
- The first derivative of W is smooth.

Smoothed Particle Hydrodynamics (SPH)

- In the SPH method , the kernel approximation

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

is approximated by imagining that the matter m is divided into N small volume elements with masses $m_1, m_2, \dots, m_n$ (nodal quadrature/particle approximation).

- The contribution to the integral from the volume element I with mass m_I and centre of mass x_I is:

$$W_h(x-x_I)u(x_I)\Delta V_I$$

where

$$\Delta V_I = \frac{m_I}{\rho(x_I)}$$

$\rho(x_I)$ = the density at node x_I

Smoothed Particle Hydrodynamics (SPH)

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

$$u_h(x) = \sum_{I \in Q(x)} W_h(x-x_I)u(x_I)\Delta V_I$$

$Q(x)$ = the set of nodes in the domain of influence of x (the set of x_I nodes)

$$u_h(x) = \sum_{I \in Q(x)} \phi_I(x)u(x_I)$$

$$\phi_I(x) = W_h(x-x_I)\Delta V_I \quad (\text{SPH shape function})$$

Smoothed Particle Hydrodynamics (SPH)

1. It is difficult to automatically calculate ΔV_I for many engineering problems without using a mesh.
2. 3D domain partitioning can even be more difficult.
3. When the initial particles are defined, the subsequent updating is handled by SPH, and can simulate many extreme situations, such as explosion and penetration.
4. The SPH shape functions do not satisfy the kronecker delta function. As a result, in general

$$u_I \neq u_h(x_I)$$

u_I is a nodal parameter at node I ,
but is not a nodal value of the field parameter at node I

Smoothed Particle Hydrodynamics (SPH)

5. SPH is an approximation of a function, not an interpolation of a function. The approximated function does not pass through the parameter values at the nodes used for constructing the shape functions.
6. The true value of the field variable at a node should be retrieved using the SPH approximation again,
 1. Obtaining the nodal parameters u_I at all the field nodes (particles)
 2. Calculating the approximated function at point $x = x_I$

Derivative Approximation by SPH

- SPH approximation for the field variable u

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

- Instead of direct computation of derivative of u ($u_{,x}$)

$$\frac{d}{dx}[u_h(x)] = \frac{d}{dx} \left[\int_{\Omega} W_h(x-y)u(y)dy \right]$$

- The same SPH analogy is also valid for the derivative of u ($u_{,x}$)

$$u_h(x) \Rightarrow u_{h,x}(x) \quad \therefore$$

$$u_{h,x}(x) = \int_{\Omega} W_h(x-y)u_{,x}(y)dy$$

$$u_{h,x}(x) = - \int_{\Omega} W_{h,x}(x-y)u(y)dy$$

Derivative Approximation by SPH

- SPH approximation for the field variable u

$$u_h(x) = \int_{\Omega} W_h(x-y)u(y)dy$$

- The same analogy is valid for the spatial derivative of u

$$\nabla.u_h(x) = \int_{\Omega} W_h(x-y)\nabla.u(y)dy$$

- After some manipulation,

$$\nabla.u_h(x) = -\int_{\Omega} u(y).\nabla W_h(x-y)dy$$

- Or in its discrete form,

$$\nabla.u_h(x) = -\sum_{I \in \Omega(x)} u(x_I)\nabla W_h(x-x_I)\Delta V_I$$

Approximation by SPH

$$u_h(x) = \sum_{I \in Q(x)} W_h(x - x_I) u(x_I) \Delta V_I$$

$$\nabla \cdot u_h(x) = - \sum_{I \in \Omega(x)} u(x_I) \nabla W_h(x - x_I) \Delta V_I$$

1. SPH approximations are used to solve a boundary value problem by directly applying the PDE at the nodes x_I .
2. The major disadvantage of the SPH is its poor accuracy.
3. A large number of nodes are required to achieve reasonable accuracy in practical applications.
4. SPH is equivalent to the Shepard interpolant.

Approximation by SPH

5. An analytical kernel has the advantage of evaluating the kernel function at any spatial point without knowing the local particle distribution.
6. The above, is not valid for the corrective SPH methods, because the corrective kernel function depends on the local particle distribution.
7. SPH kernel functions satisfy 0-th order moment condition.
8. Most SPH kernel functions satisfy higher order moment conditions in the continuous form
9. In contrary, discretized forms of SPH do not satisfy higher order moment conditions

SPH versus Shepard PU

- Shepard interpolant:

$$u_h(y) = \sum_{I=1}^N w_I(y) u(x_I)$$

$$w_I(y) = \frac{W_I(y)}{\sum_{J=1}^N W_J(y)}, W_I(y) = \text{Weighting function}$$

- SPH approximation

$$W_h(x - x_I) = \frac{W_I(x)}{\Delta V_I \sum_{J=1}^N W_J(x)} \quad (\text{new SPH kernel})$$

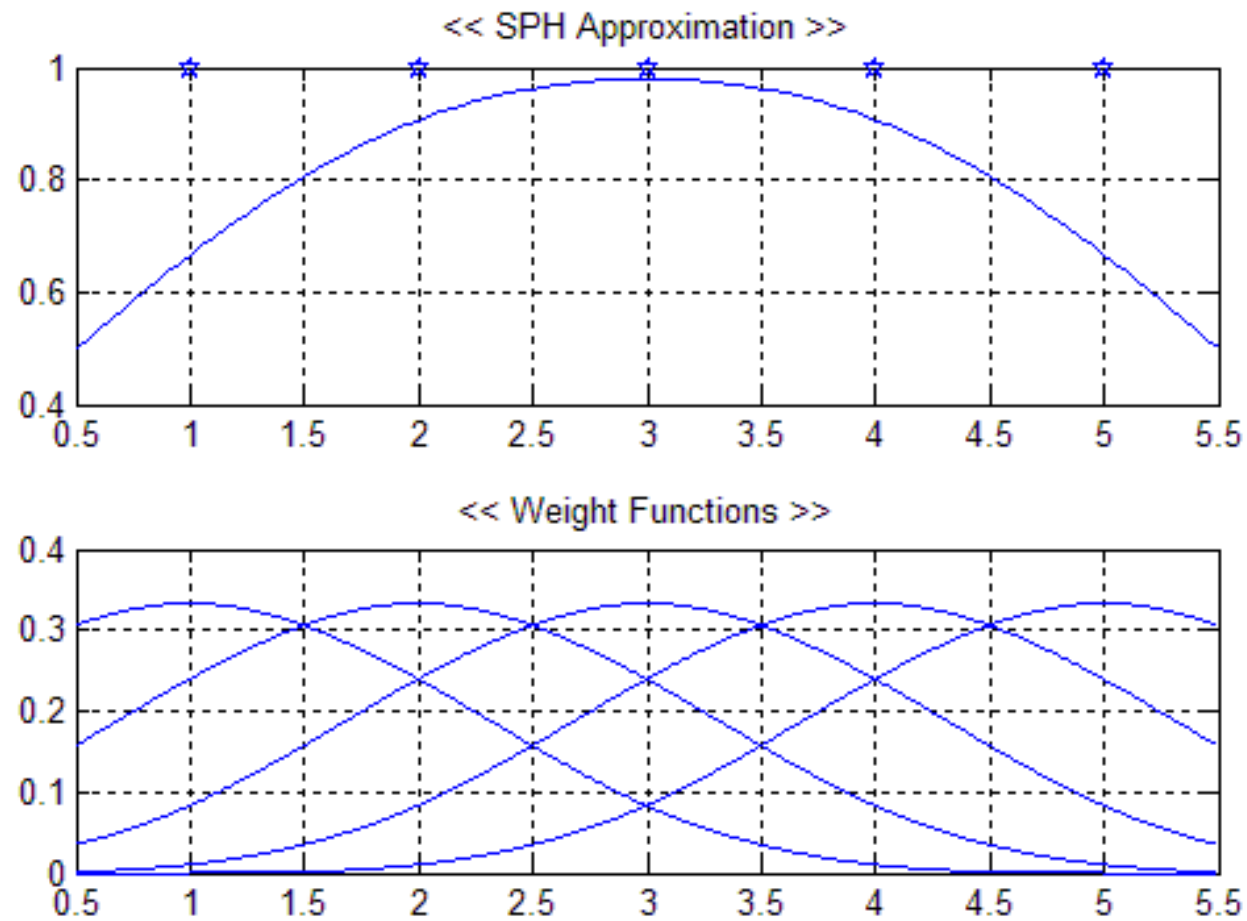
$$u_h(x) = \sum_I W_h(x - x_I) u(x_I) \Delta V_I = \sum_{I=1}^N w_I(x) u(x_I)$$

SPH Approximation of a Constant Function

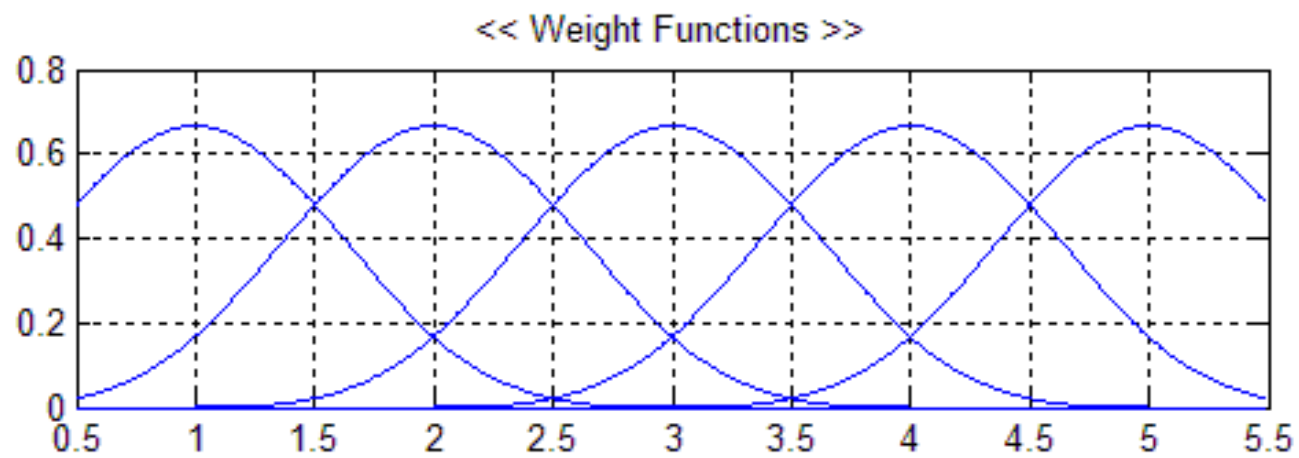
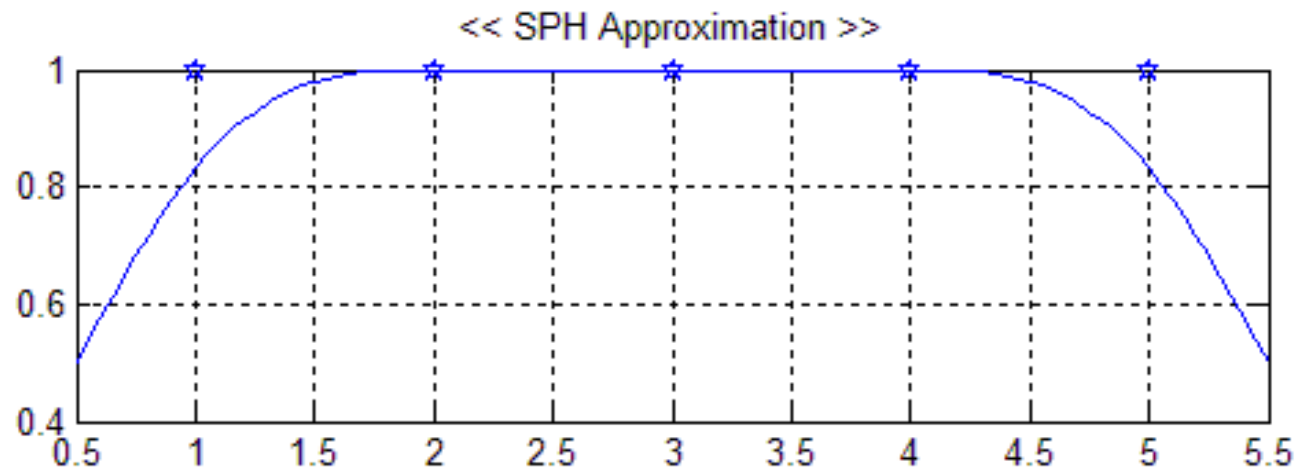
- Constant function $f(x)=1$
- Five equally distributed points
- Cubic spline weight function

$$W(r, h) = \begin{cases} \frac{2}{3h} \left[1 - \frac{3}{2}q^2 + \frac{3}{4}q^3 \right] & q \leq 1 \\ \frac{1}{4h} [2 - q]^3 & 1 \leq q < 2 \\ 0 & q \geq 2 \end{cases}$$

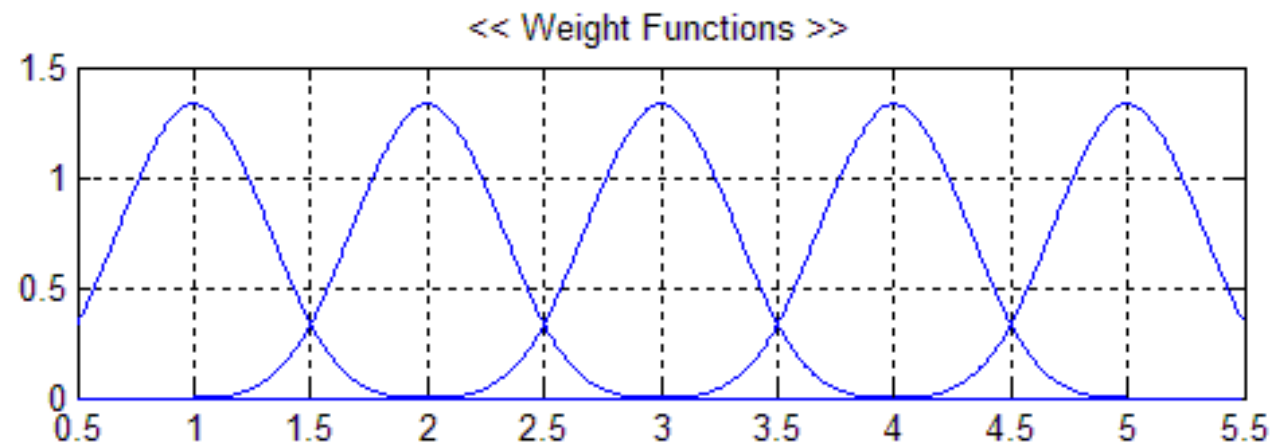
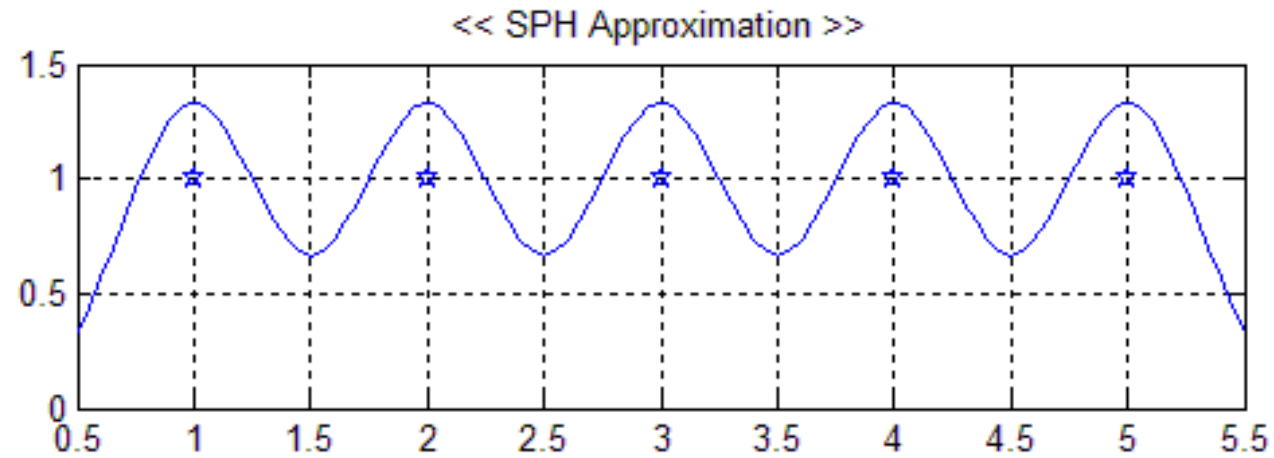
$$q = \frac{r}{h}, \quad r = |x - x_I|$$



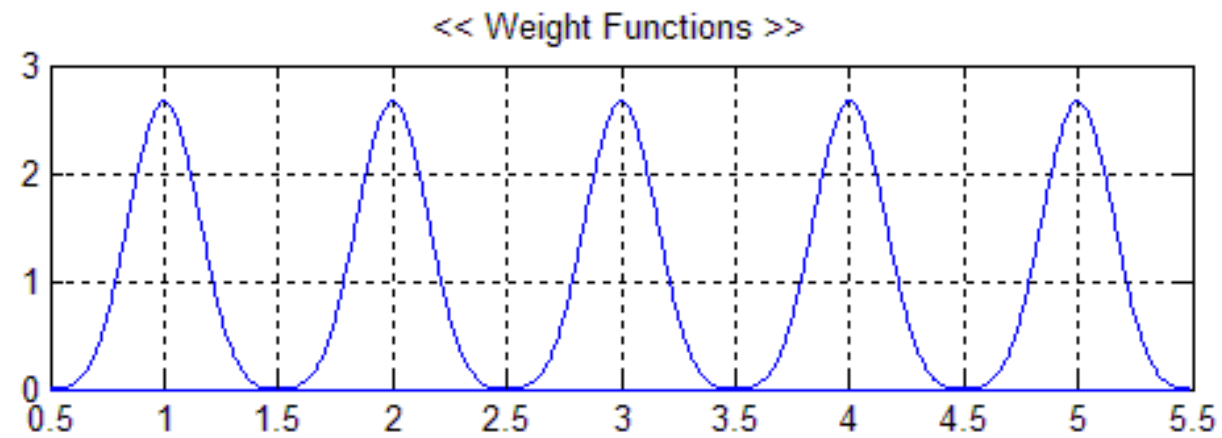
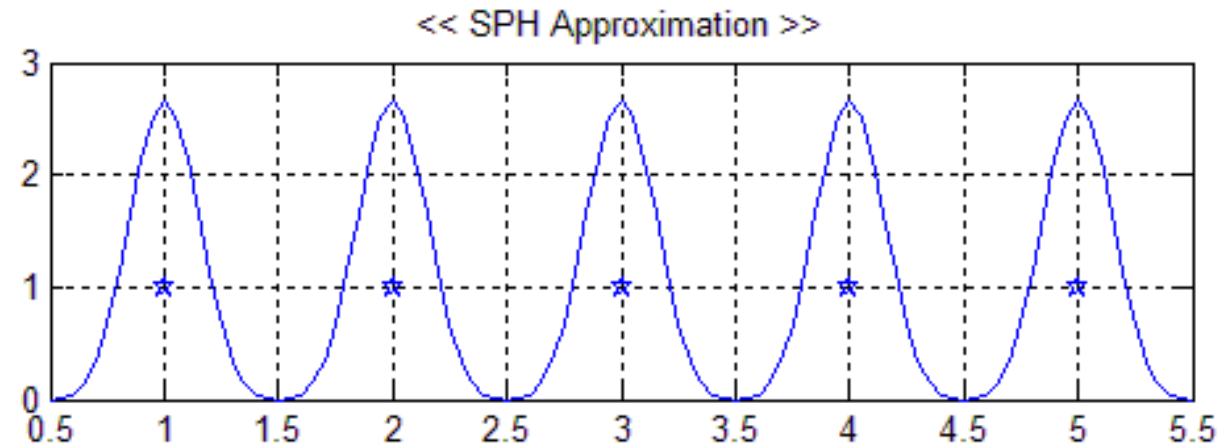
$f(x)=1$, W = cubic spline, $h=2.0$



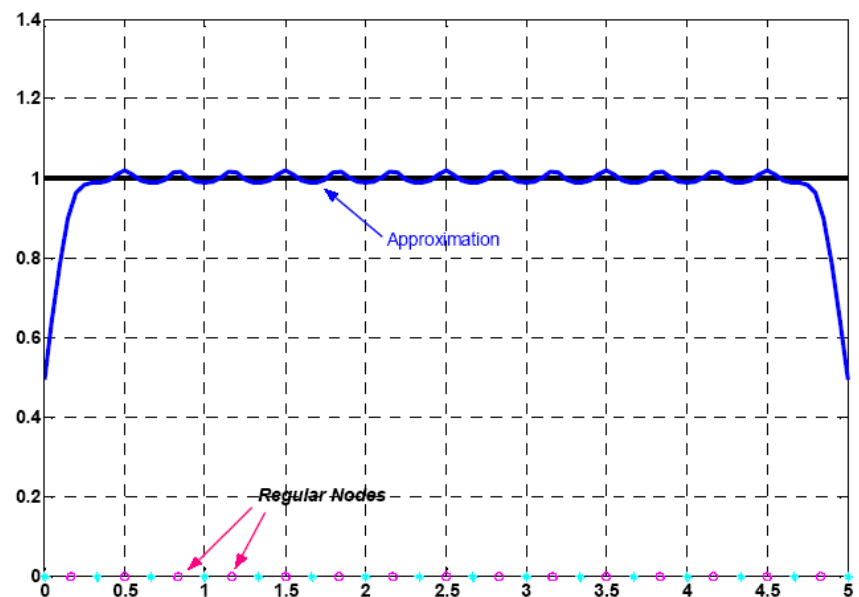
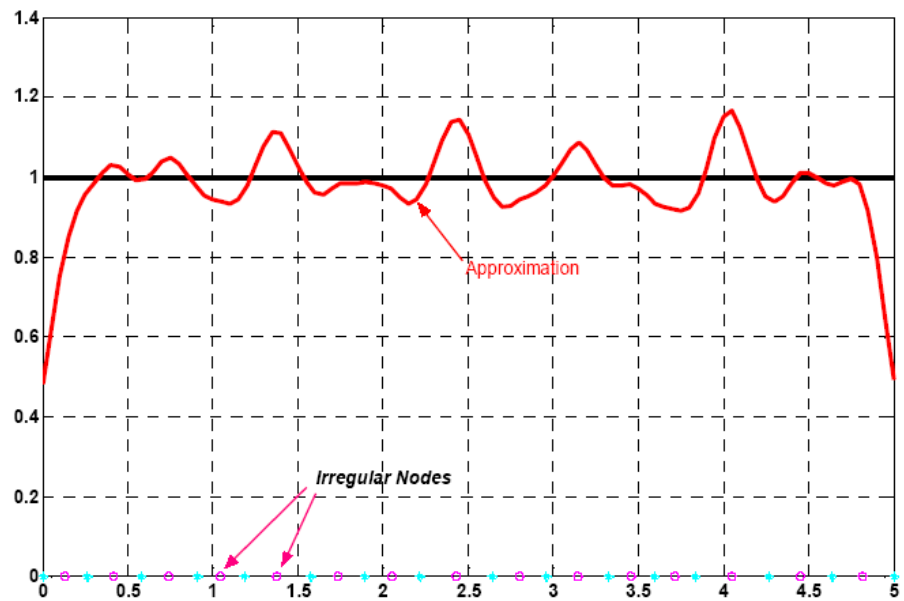
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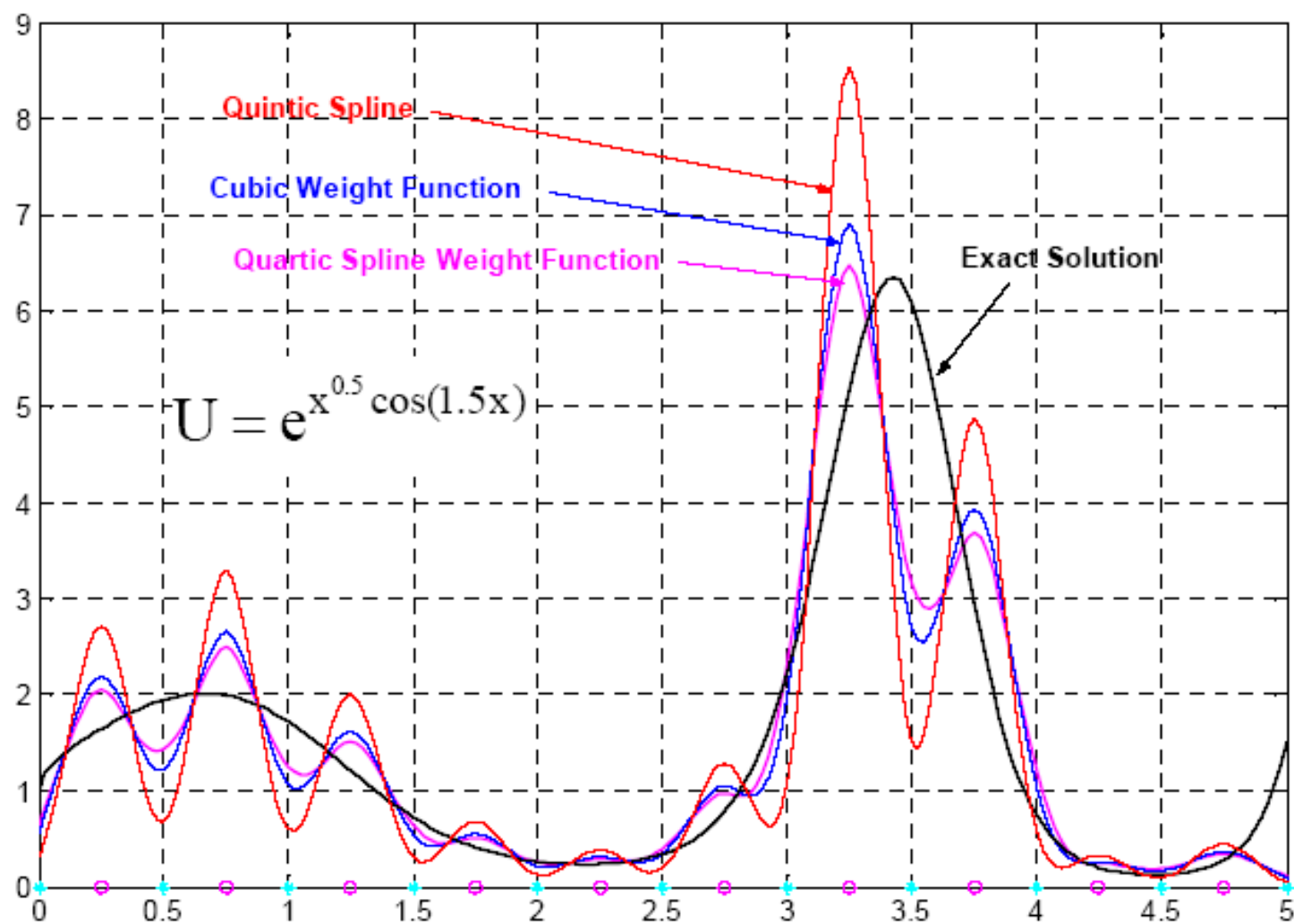


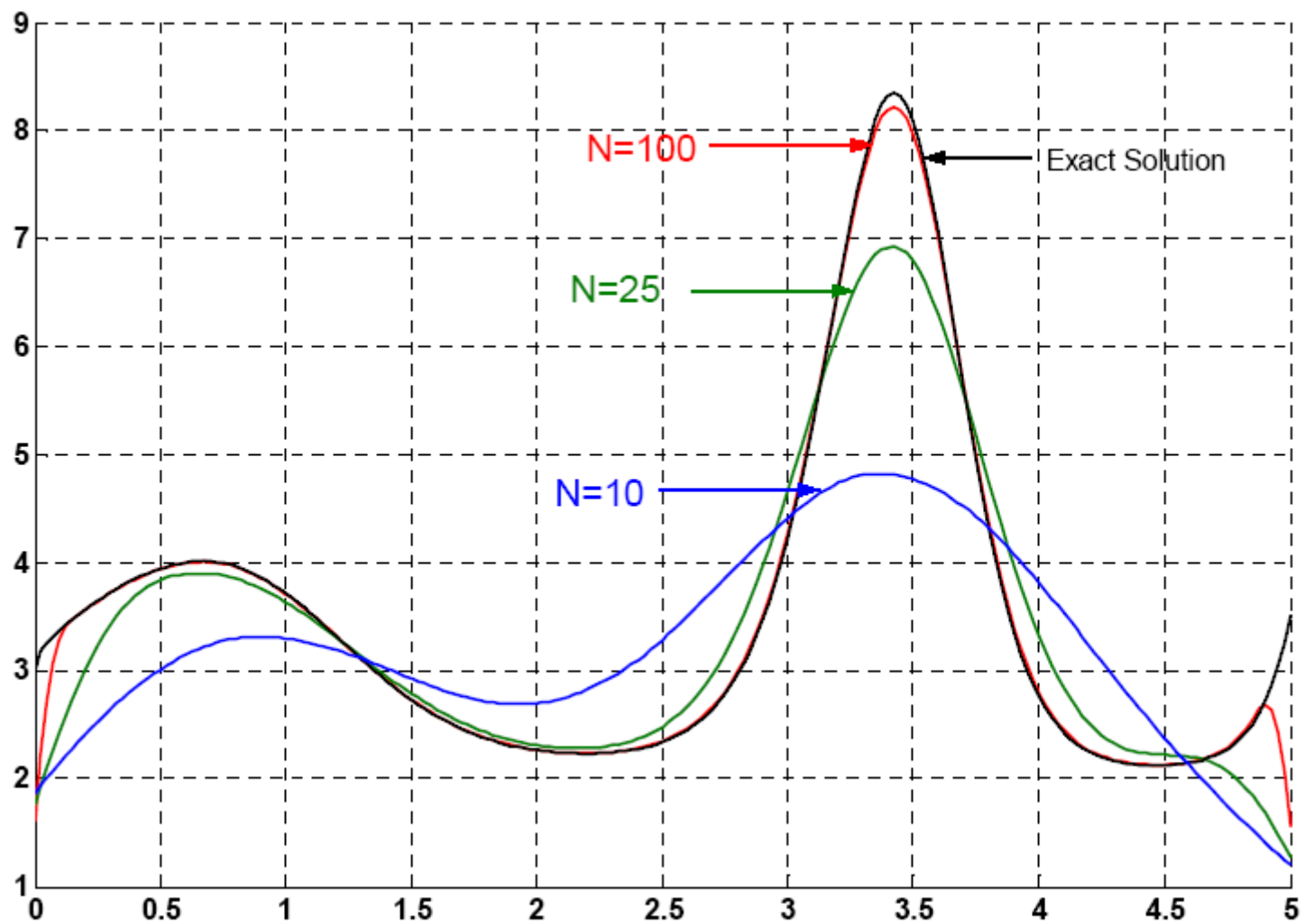
$f(x)=1$, W = cubic spline, $h=0.5$

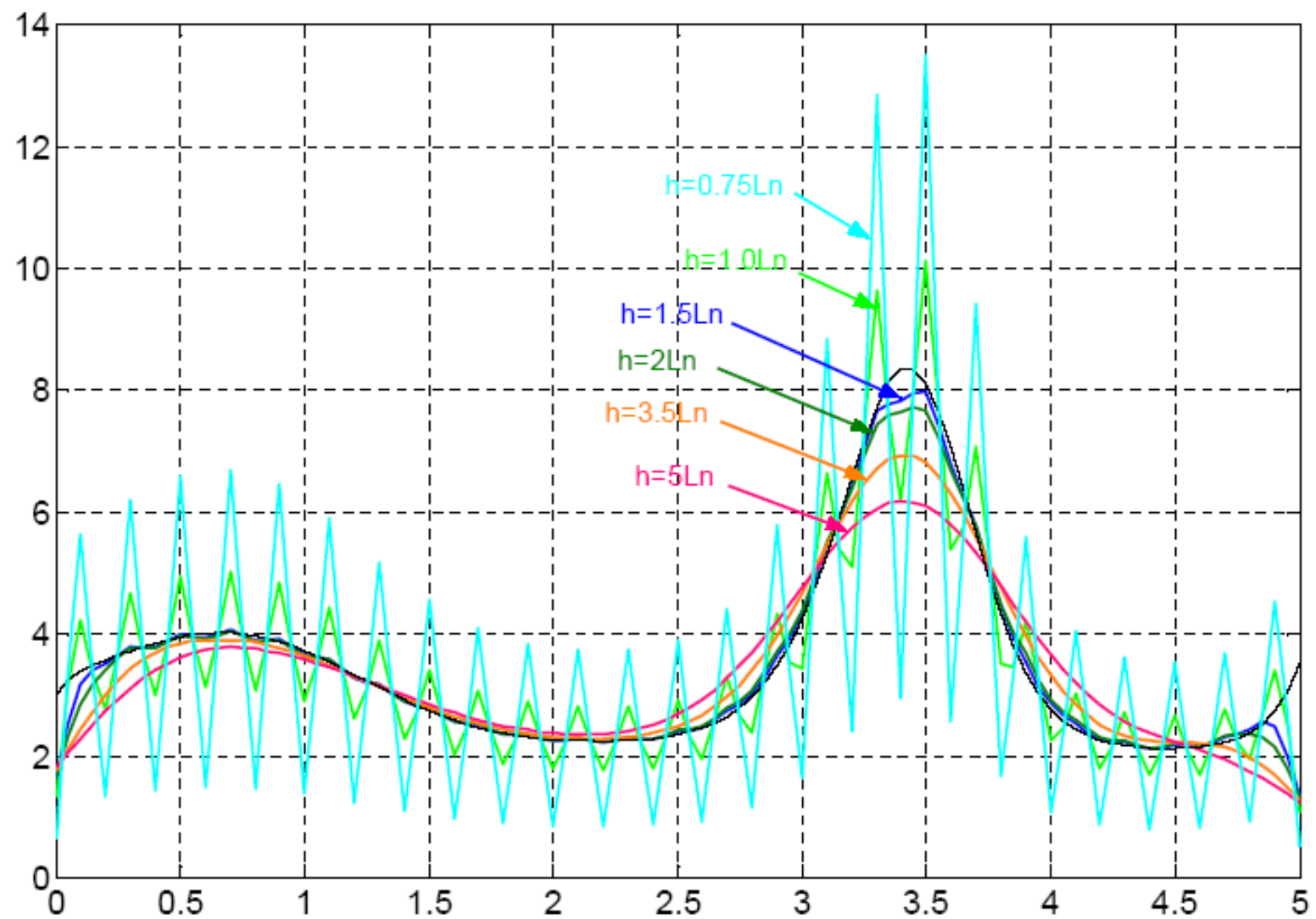


$f(x)=1$, W = cubic spline, $h=0.25$









Disadvantages of SPH

1. Tensile instability: unstable particles motion when subjected to certain tensile/hydrostatic stress state.
2. Lack of interpolation consistency (completeness).
3. Zero-energy modes.
4. Difficulty in enforcing essential boundary conditions.
5. SPH is not a partition of unity. As a result, SPH interpolants can not represent rigid body motion correctly.

Tensile Instability

- For particles under certain tensile (hydrostatic) stress state, the motion of the particles may become unstable.
- Apparently, the SPH tensile instability is related to the lack of consistency of the SPH interpolant.
- Under certain tensile pressure conditions, some particles may form clusters of particles (instead of moving outward from each other)

Remedies for tensile instability

1. Morris (1996)

- Proposed using special kernel functions.
- They do not always yield satisfactory results.

2. Randles-Libersky (1996)

- Adding dissipative terms, which is related to conservative smoothing.

3. Liu-Belytschko (1995-1998)

- Reproducing kernel particle method (RKPM)

4. Dilts (1997)

- Replacing the standard SPH interpolant with a more sophisticated MLS interpolant

Remedies for tensile instability

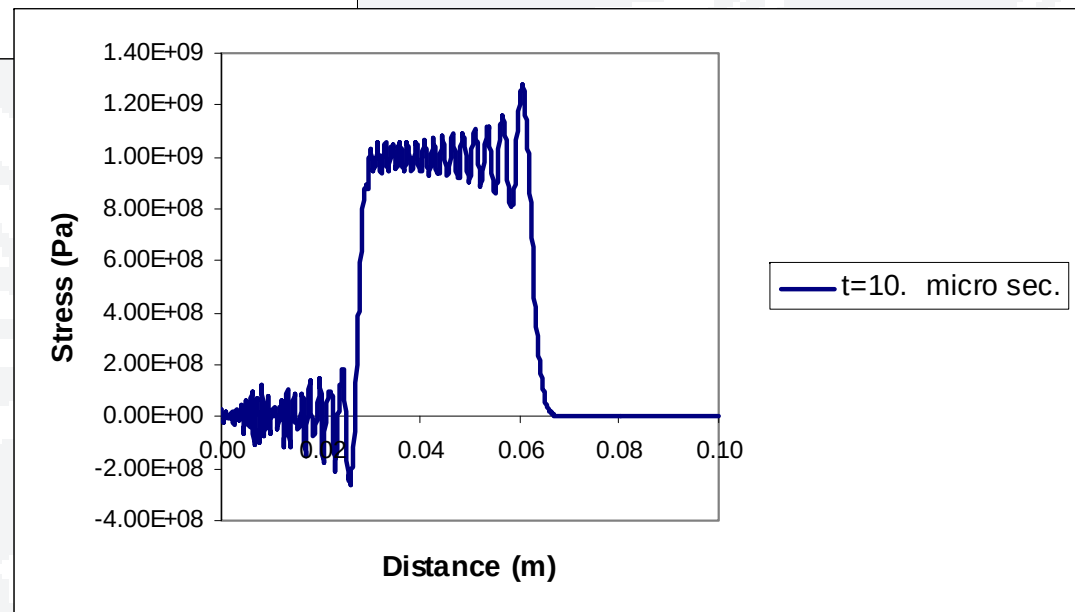
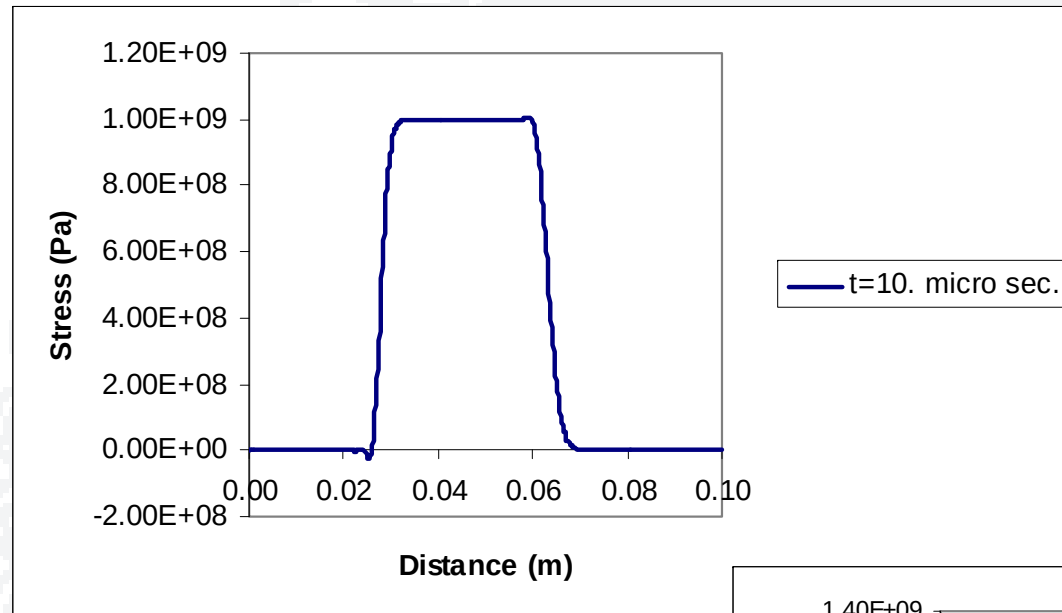
5. Dyka-Ingle (1995)

- Also called stress point method.
- The essential idea is to add additional points other than SPH particles when evaluating/sampling stress and other state variables, while kinematic variables (such as velocity and acceleration) are still sampled at particle points.

6. Chen-Beraun (2000)

- Generalized smoothed particle hydrodynamics.

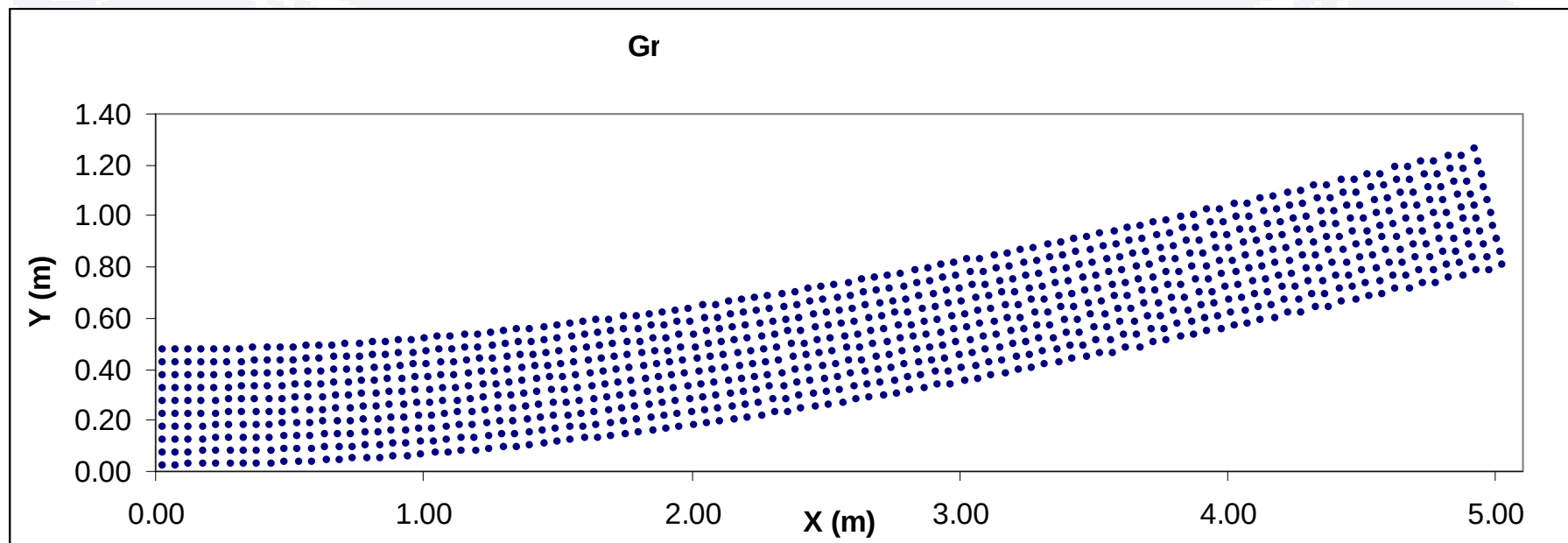
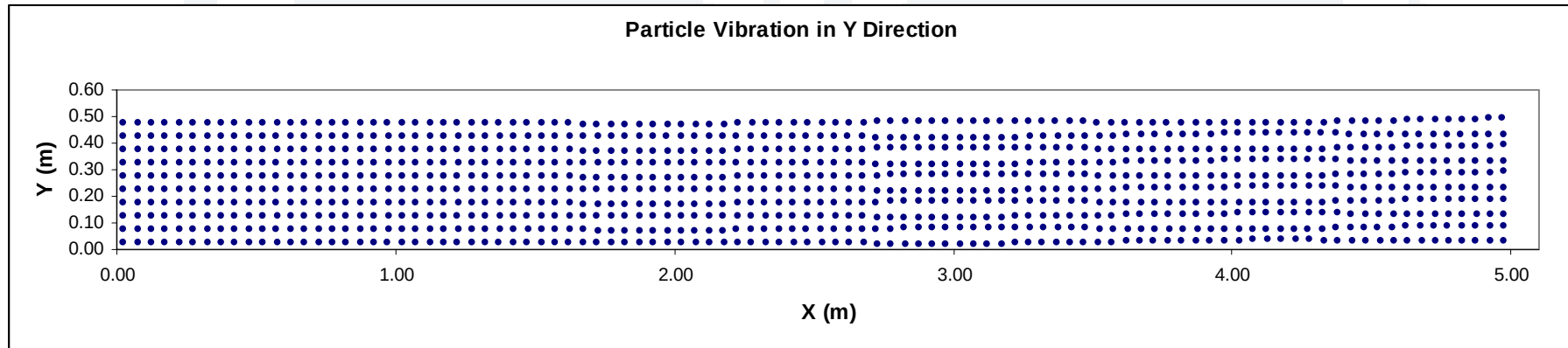
Oscillations: Axial Impact



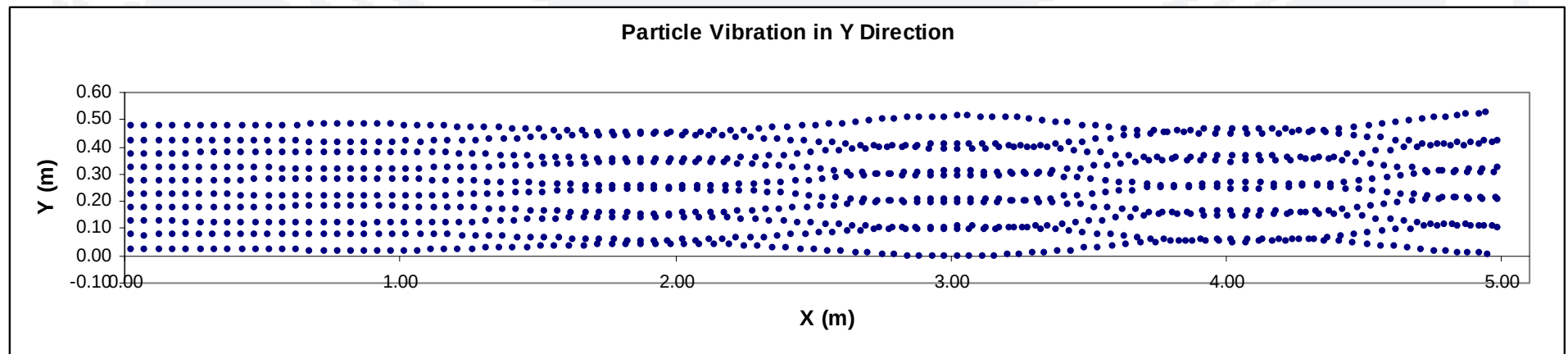
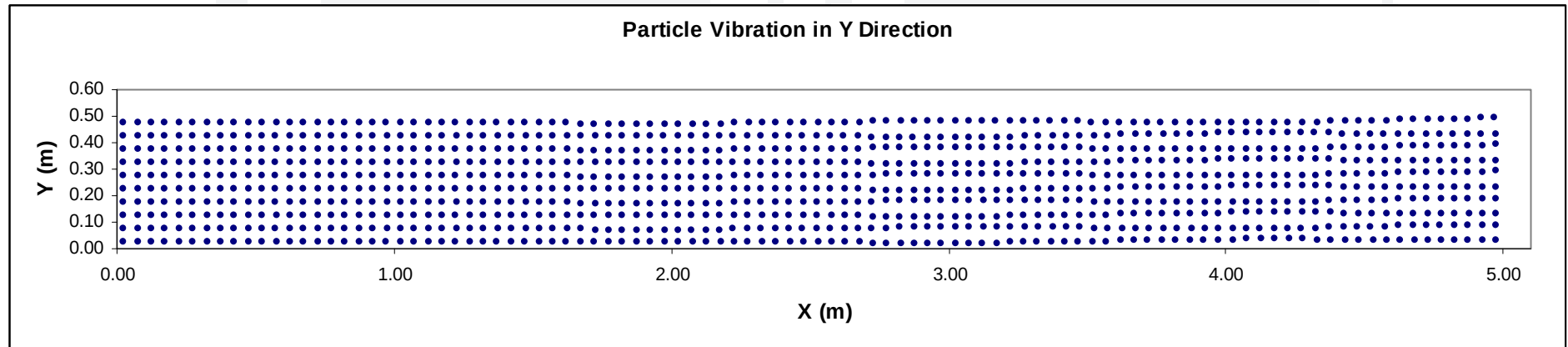
Zero Energy Modes

- The reason is due to the fact that the derivatives of kinematic variables are evaluated at particle points by analytical differentiation rather than by differentiation of interpolants.
- In many cases, the kernel function reaches a maximum at its nodal position, and its spatial derivatives become zero.
- An efficient remedy is to adopt the stress point approach.

Zero Energy Modes



Zero Energy Modes



Boundary Conditions

- SPH and other particle methods have difficulties in enforcing essential boundary conditions.

- Consider 1D constant function $f(x) = c$.

- For points I away from boundaries: $\int W dx = 1$

$$f_I = \int f W dx = \int c W dx = c \int W dx = c$$

- For points I locating at boundaries: $\int W dx = 0.5$

$$f_I = \int f W dx = \int c W dx = c \int W dx = \frac{1}{2} c$$

- For points I near a boundary:

$$\frac{1}{2} c < f_I < c$$

Remedies for Boundary Conditions

1. Takeda-Miyama-Sekiya (1994)

- Developed Image Particle Method (IPM) to satisfy the no-slip boundary conditions.

2. Morris-Fox-Zhu (1997)

- Generalized IPM to satisfy boundary conditions along a curved boundary.

3. Johnson-Beissel (1996)

- Normalized kernel function to improve the strain calculations.

Remedies for Boundary Conditions

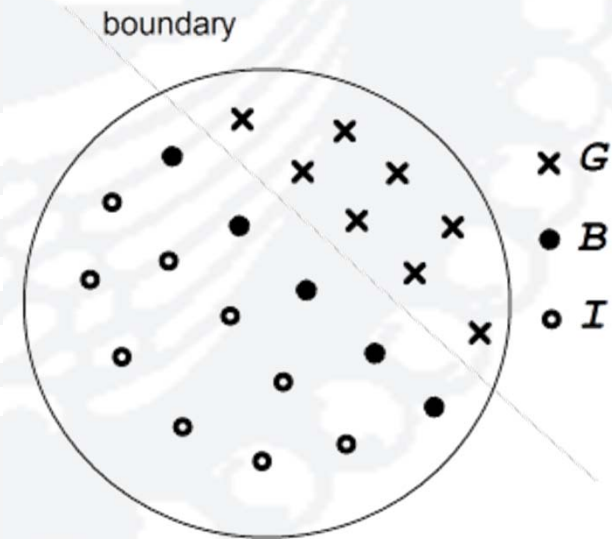
4. Randles-Libersky (1996)

- Ghost Particle Approach (GPA) based on the same philosophy of generalized image particle method.
- In this method, for a boundary particle I , all other particles within its support domain $\Omega(I)$ can be divided into

- $\mathbf{I}(I)$: all the neighbor interior points
- $\mathbf{B}(I)$: all the neighbor boundary points
- x $\mathbf{G}(I)$: all the neighbor exterior points

(Ghost Particles)

$$\Omega(I) = \mathbf{I}(I) \cup \mathbf{B}(I) \cup \mathbf{G}(I)$$



Remedies for Boundary Conditions

- The boundary correction formula for general scalar field f at boundary node $I(f_I)$ is given by

$$f_I = \bar{f}_I + \frac{\sum_{J \in \mathbf{I}(I)} (f_J - \bar{f}_J) \Delta V_J W_{IJ}}{(1 - \sum_{J \in \mathbf{B}(I)} \Delta V_J W_{IJ})}$$

$\bar{f}_I$ prescribed boundary value at $x = x_I$

- Note that the sampling formula only depends on interior particles

Consistency

- The solution must approach the exact solution when the nodal (particle) spacing approaches to zero.
- $C^k = k^{th}$ order consistency:
if the approximation can produce a polynomial of up to k^{th} order exactly (also termed completeness)
- For a PDE of order $2k$, the minimum requirement of the consistency for Galerkin based formulations is C^k .
- Reproducing a polynomial $\equiv$ representing a polynomial exactly

Consistency

- C^0 consistency:

$$\int_{\Omega} W(x-y, h) dy = 1 \quad (\text{condition of unity})$$

- C^1 consistency:

$$x = \int_{\Omega} y W(x-y, h) dy$$

or,

$$\int_{\Omega} (x-y) W(x-y, h) dy = 0 \quad (\text{the first moment of } W)$$

or similarly,

$$\int_{\Omega} x W(x, h) dx = 0$$

Approximation by SPH

- Discretized forms of SPH do not satisfy moment conditions,

$$\sum_{I=1}^N W(x - x_I, h) \Delta x_I \neq 1$$

$$\sum_{I=1}^N (x - x_I) W(x - x_I, h) \Delta x_I \neq 0$$

- The first moment condition is valid only if the weight function is symmetric about the origin x .
- The first moment condition is difficult (if not impossible) to be met on and near the boundary.

Modified SPH Methods:

1. Reproducing Kernel Particle Method (RKPM)
2. Monaghan's symmetrization on derivative approximation
3. Johnson-Beissel correction
4. Randles-Libersky correction
5. Krongauz-Belytschko correction
6. Chen-Beraun correction
7. Bonet-Kulasegaram integration correction
8. Aluru collocation RKPM
9. ...

Modified SPH Methods:

1. The linear reproducing condition in the interpolation is equivalent to the constant reproducing condition in the derivative of the interpolant.
2. As a result, some of the algorithms directly correct derivatives instead of the interpolant.
3. The Chen-Beraun correction corrects higher order derivatives, but it may require more computational effort in multi-dimensions.

Corrective SPH (CSPM):

Reproducing Kernel Particle Methods (RKPM)

- The idea is to construct a corrective kernel, a product of the correction function C with the original kernel W , to ensure consistency/completeness of the SPH interpolant.

$$\bar{W}(x - x_I, x) = C_h(x - x_I, x)W(x - x_I, x)$$

$$C_h(x - x_I, x) = \text{the correction function}$$

- $C_h(x - x_I, x)$ is designed to correct the original kernel function to satisfy a required order of consistency.
- CSPM is expected to improve the approximation near the boundaries.

RKPM – CSPM :

$$\overline{W}(x - x_I, x) = C_h(x - x_I, x)W(x - x_I, x)$$

- C_h can be expressed as,

$$C_h(x - x_I, x) = b_0(x, h) + b_1(x, h)\frac{x - x_I}{h} + b_2(x, h)\left(\frac{x - x_I}{h}\right)^2 + \dots$$

$b_0(x), b_1(x), \dots, b_n(x)$ are unknown functions, determined to correct the original kernel function

- *Now, suppose $f(x)$ is a sufficiently smooth function,*

$$f_I = f(x_I) = f(x) + f'(x)\left(\frac{x_I - x}{h}\right)h + \frac{f''(x)}{2!}\left(\frac{x_I - x}{h}\right)^2 h^2 + \dots$$

RKPM – CSPM :

- The modified kernel approximation can be written as,

$$f^h(x) = \sum_{I=1}^{NP} \bar{W}_h(x - x_I, x) f_I \Delta x_I$$

$$\begin{aligned} f^h(x) = & \left(\sum_{I=1}^{NP} \bar{W}_h(x - x_I, x) \Delta x_I \right) f(x) h^0 \\ & - \left(\sum_{I=1}^{NP} \left(\frac{x - x_I}{h} \right) \bar{W}_h(x - x_I, x) \Delta x_I \right) f'(x) h \\ & + \dots \\ & + \left(\sum_{I=1}^{NP} (-1)^n \left(\frac{x - x_I}{h} \right)^n \bar{W}_h(x - x_I, x) \Delta x_I \right) \frac{f^{(n)}(x)}{n!} h^n + \Theta(h^{n+1}) \end{aligned}$$

RKPM – CSPM :

- To obtain an n^{th} order reproducing kernel, the moments of the modified kernel function must satisfy the following conditions:

$$M_0(x) = \sum_{I=1}^{NP} \bar{W}_h(x - x_I, x) \Delta x_I = 1$$

$$M_1(x) = \sum_{I=1}^{NP} \left(\frac{x - x_I}{h} \right) \bar{W}_h(x - x_I, x) \Delta x_I = 0$$

$\vdots$

$$M_n(x) = \sum_{I=1}^{NP} \left(\frac{x - x_I}{h} \right)^n \bar{W}_h(x - x_I, x) \Delta x_I = 0$$

RKPM – CSPM :

- Using the definitions for $\overline{W}_h$, C_h and M_n :

$$\begin{bmatrix} m_0(x) & m_1(x) & \dots & m_n(x) \\ m_1(x) & m_2(x) & \dots & m_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ m_n(x) & m_{n+1}(x) & \dots & m_{2n}(x) \end{bmatrix} \begin{bmatrix} b_0(x, h) \\ b_1(x, h) \\ \vdots \\ b_n(x, h) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$m? \rightarrow ?M$$

- After introducing the correction function C_h , the modified kernel function may not be a positive function anymore, (also called a signed partition of unity, after Duaret/Oden(1996))

$$\overline{W}_h(x - x_I) < 0$$

RKPM-1D (1995):

- Continuous form:

$$\overline{W}(x-y, x) = C_h(x, y)W(x-y, h)$$

$$u_h(x) = \int_{\Omega} u(y)C_h(x, y)W(x-y, h)dy$$

- For 1D problems:

$$C(x, y) = C_1(x) + C_2(x)(x-y)$$

- $C_1(x)$ and $C_2(x)$ are found by enforcing the corrected kernel to reproduce the function.

RKPM-1D (1995):

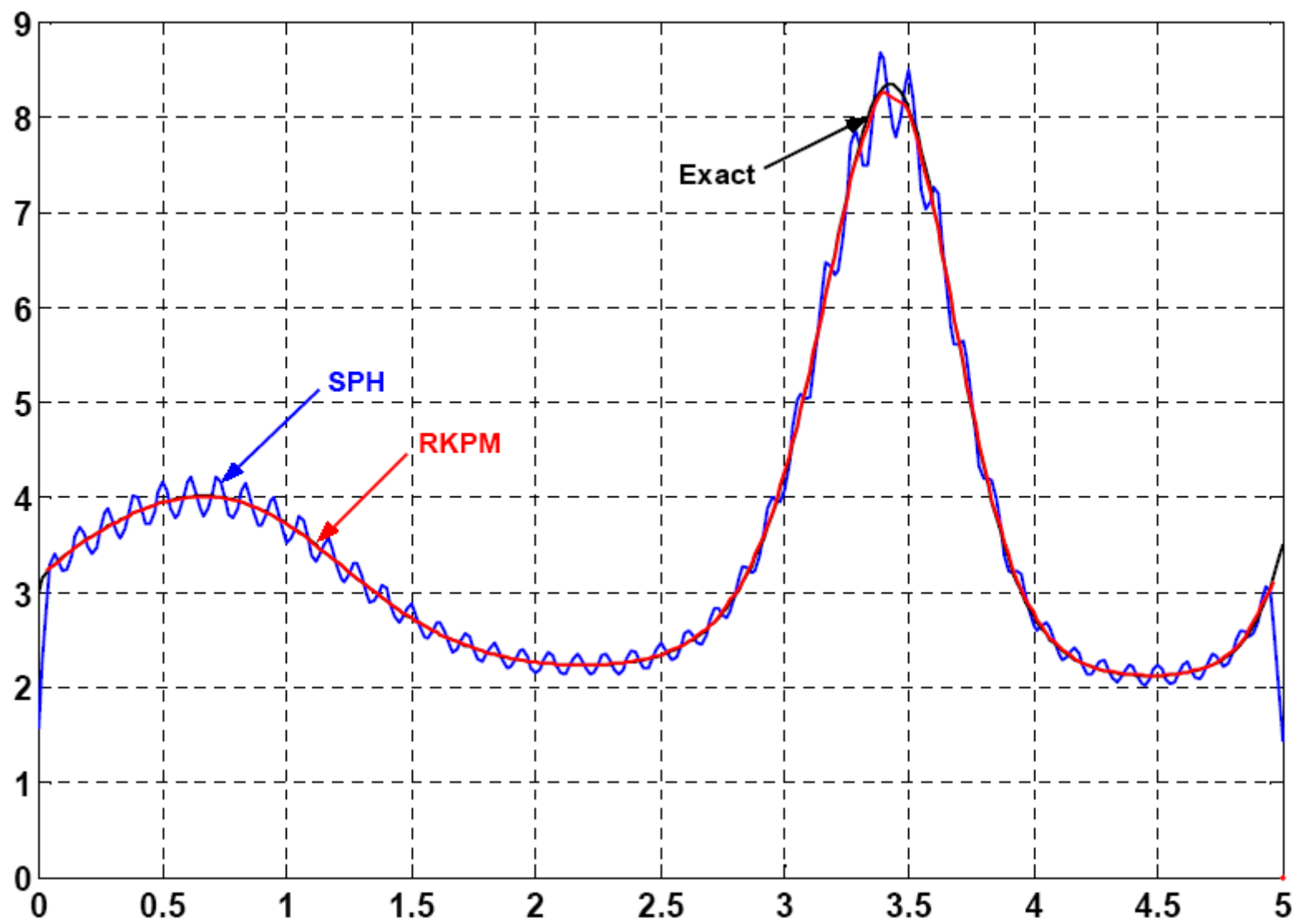
$$C_1(x) = \frac{m_2(x)}{(m_0(x)m_2(x) - m_1^2(x))}$$

$$C_2(x) = \frac{m_1(x)}{(m_0(x)m_2(x) - m_1^2(x))}$$

$$m_0(x) = \int_{\Omega} W(x-y)dy$$

$$m_1(x) = \int_{\Omega} yW(x-y)dy$$

$$m_2(x) = \int_{\Omega} y^2W(x-y)dy$$



Generalized SPH (G-SPH):

- Proposed by Chen-Beraun (2000)
- G-SPH is developed to remedy the boundary deficiencies.
- Principle steps are:
 - Expanding Taylor series for f about a point (x_p, y_p)
 - Multiplying both sides with a kernel function w
 - Integrating over the entire domain

Generalized SPH (G-SPH):

$$\begin{aligned}\int_{\Omega} f(x) W dA = & f(x_I) \int_{\Omega} W dA + \\ & f_x(x_I) \int_{\Omega} (x - x_I) W dA + f_y(x_I) \int_{\Omega} (y - y_I) W dA + \\ & \frac{1}{2} f_{xx}(x_I) \int_{\Omega} (x - x_I)^2 W dA + \\ & f_{xy}(x_I) \int_{\Omega} (x - x_I)(y - y_I) W dA + \\ & \frac{1}{2} f_{yy}(x_I) \int_{\Omega} (y - y_I)^2 W dA + \dots\end{aligned}$$

$$f_{\alpha} = \frac{\partial f}{\partial \alpha} \quad , \quad f_{\alpha\beta} = \frac{\partial^2 f}{\partial \alpha \partial \beta} \quad , \quad \alpha, \beta = x, y$$

Generalized SPH (G-SPH):

- As a first approximation, all the derivative terms are neglected and a corrective version of the kernel is generated:

$$f(x_I) = \frac{\int_{\Omega} f(x) W dA}{\int_{\Omega} W dA}$$

$$f_I = \frac{\sum_{J=1}^N f_J W_{IJ} \frac{m_J}{\rho_J}}{\sum_{J=1}^N W_{IJ} \frac{m_J}{\rho_J}}$$

Generalized SPH (G-SPH):

- For a point where its domain of influence locates fully inside a problem domain, $(x-x_I)W$ and $(y-y_I)W$ vanish due to the symmetry of W .
- As a result, the Taylor series truncation error is of order $h_I^2(\Delta x_I^2 \text{ and } \Delta y_I^2)$.
- For points near or on boundaries, the truncation error becomes of order $h_I(\Delta x_I \text{ and } \Delta y_I)$, because $(x-x_I)W$ and $(y-y_I)W$ are non-zero.
- In case of 1-D constant function $f=C$, (for both interior and on boundary points)

$$f_I = \frac{\int_{\Omega} f(x)WdA}{\int_{\Omega} WdA} = \frac{\int_{\Omega} CWdA}{\int_{\Omega} WdA} = \frac{C \int_{\Omega} WdA}{\int_{\Omega} WdA} = C$$

1-D Derivatives (G-SPH):

- The generalized derivative approximations in 1D can be constructed by simply neglecting the corresponding higher-order derivative terms of the Taylor series,

$$f_x(x_I) = f_{xI} = \frac{\int_{\Omega} [f(x) - f_I] W dx}{\int_{\Omega} (x - x_I) W dx}$$

$$f_{xx}(x_I) = f_{xxI} = \frac{\int_{\Omega} [f(x) - f_I] W dx - f_{xI} \int_{\Omega} [x - x_I] W dx}{\frac{1}{2} \int_{\Omega} (x - x_I)^2 W dx}$$

1-D Derivatives (G-SPH):

1. In standard SPH, derivative estimates are derived using integration by parts, so the derivatives of W are involved. While in G-SPH, it is not necessary to use the derivatives of W as a kernel for the generalized derivative estimate.
2. Denominators of 1-D G-SPH derivatives must be non-zero,
 - For evaluating f_x :
 - Any anti-symmetric W function can be chosen for evaluating f_x .
 - The anti-symmetric $W_{,x}$ is one of the selections for the kernel function of f_x . $W_{,xx}$ is then used for f_{xx} based on f_x .
 - In general, W gives better results than $W_{,xx}$.
 - Any Symmetric W function can be employed for directly evaluating $f_{,xx}$.

1-D Derivatives (G-SPH):

3. W does not have the properties of the computational kernel functions and may not be called a kernel function.
4. Consistency
 - f_x exactly represent constant fields.
 - $f_{,xI}$ exactly represent constant gradients.
 - $f_{,xxI}$ exactly represent linear gradients.
5. The total volume of particles involved in the computation of G-SPH is closer to that confined by $2h_p$.

